

A HILBERT–MUMFORD CRITERION FOR POLYSTABILITY IN KAEHLER GEOMETRY

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ABSTRACT. Consider a Hamiltonian action of a compact Lie group K on a Kaehler manifold X with moment map $\mu : X \rightarrow \mathfrak{k}^*$. Assume that the action of K preserves the complex structure of X , and consider its unique extension to a holomorphic action of the complexification G of K . We characterize which G -orbits in X intersect $\mu^{-1}(0)$ in terms of the maximal weights $\lim_{t \rightarrow \infty} \langle \mu(e^{its} \cdot x), s \rangle$, where $s \in \mathfrak{k}$. We do not impose any a priori restriction on the stabilizer of x . Under some mild restrictions on the growth of μ and the action $K \curvearrowright X$, we view the maximal weights as defining a collection of maps, for each $x \in X$,

$$\lambda_x : \partial_\infty(K \backslash G) \rightarrow \mathbb{R} \cup \{\infty\},$$

where $\partial_\infty(K \backslash G)$ is the boundary at infinity of the symmetric space $K \backslash G$. We prove that $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$ if: (1) λ_x is everywhere nonnegative, (2) any boundary point y such that $\lambda_x(y) = 0$ can be connected with a geodesic in $K \backslash G$ to another boundary point y' satisfying $\lambda_x(y') = 0$. We also prove that the maximal weight functions are G -equivariant: for any $g \in G$ and any $y \in \partial_\infty(K \backslash G)$ we have $\lambda_{g \cdot x}(y) = \lambda_x(y \cdot g)$.

1. INTRODUCTION

Let K be a compact connected Lie group with Lie algebra \mathfrak{k} , let X be a (non necessarily compact) Kaehler manifold, and assume that there is an action $K \curvearrowright X$ by holomorphic isometries (in particular, preserving the symplectic form) and admitting a moment map $\mu : X \rightarrow \mathfrak{k}^*$. Since μ is by definition K -equivariant, the action of K on X preserves the zero level set $\mu^{-1}(0)$ and the quotient $\mu^{-1}(0)/K$ carries a natural structure of stratified symplectic manifold, see [SL]. Let G be the complexification of K . By a theorem of Guillemin and Sternberg [GS], the action of K on X extends to a unique action of G on X such that the map $G \times X \ni (g, x) \mapsto g \cdot x \in X$ is holomorphic. The action of G , however, no longer preserves the symplectic form of X nor the zero level set $\mu^{-1}(0)$.

A very natural and important question, which has been extensively treated in the literature, is to find a good notion of quotient of X by the action of G , carrying a structure of (possibly singular) Kaehler manifold induced in some way from the structure in X . The most naive possibility, taking the space of orbits X/G with the quotient topology, will not even be Hausdorff in general, so there is no hope to provide it with a structure of singular Kaehler manifold. To avoid this pathology one can restrict the attention to a big G -invariant subset $X^* \subset X$, obtained after removing some *bad* G -orbits in X , such

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that the quotient space X^*/G is Hausdorff. This faces us with the problem of defining X^* in a natural way, satisfying the previous condition and at the same time being as big as possible (for example, we would like X^* to be dense in X).

A systematic way of defining X^* uses the moment map of the action of K . (For the results stated in this paragraph and the next one, see the survey [HH] and the references therein.) One says that $x \in X$ is semistable if the closure of $G \cdot x$ intersects $\mu^{-1}(0)$. Then the set $X^{ss} \subset X$ of semistable points is open and the relation which identifies two orbits in X^{ss} if their closures intersect is an equivalence relation. The quotient space $X//G$ of this equivalence relation carries a natural structure of (possibly singular) holomorphic space, and the projection $X^{ss} \rightarrow X//G$ is holomorphic. On the other hand, if one defines $X^{ps} \subset X$ as the set of points whose G -orbit intersects $\mu^{-1}(0)$ (we call such points polystable), then the inclusion $X^{ps} \subset X^{ss}$ induces a homeomorphism $X^{ps}/G \simeq X//G$, so that one can take X^{ps} as a good choice for X^* . This motivates the following question.

Question 1.1. *Which G -orbits $\mathcal{O} = G \cdot x \subset X$ intersect $\mu^{-1}(0)$? If \mathcal{O} is such an orbit, how many K -orbits does $\mathcal{O} \cap \mu^{-1}(0)$ contain?*

In the first question we would like some characterization of the points $x \in X$ such that $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$ in terms of the symplectic geometry of the action $K \curvearrowright X$. The answer to the second question (namely, that $\mathcal{O} \cap \mu^{-1}(0)$ contains at most one K -orbit) is well known and follows from an easy argument which we recall in Section 4.1 below. A refinement of this is the statement that $X//G$ is homeomorphic to the symplectic quotient $\mu^{-1}(0)/K$, and that the holomorphic structure on $X//G$ defined in [HH] is compatible with it, in the sense that they combine to define a structure of stratified Kaehler manifold (see [HH, S]).

A partial characterization of the G -orbits intersecting $\mu^{-1}(0)$ was given in [M] in terms of the maximal weights $\lambda(x; s)$, defined for any $s \in \mathfrak{k}$ to be

$$\lambda(x; s) = \lim_{t \rightarrow \infty} \langle \mu(e^{its} \cdot x), s \rangle \in \mathbb{R} \cup \{\infty\}$$

(this limit exists, see Section 3.2 below). A point $x \in X$ was defined to be analytically stable if $\lambda(x; s) > 0$ for any nonzero $s \in \mathfrak{k}$, and [M, Theorem 5.4] states that x is analytically stable if and only if $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$ and the stabilizer G_x is finite. Apart from the restriction to points with finite stabilizers, the point of view in [M] has the inconvenient that it does not seem to allow any straightforward proof that if x is analytically stable then $g \cdot x$ is also analytically stable for any $g \in G$. Roughly speaking, this is proved in [M] as a consequence of the characterization of analytic stability in terms of the so-called linear properness of the integral of the moment map, which essentially amounts to [M, Theorem 5.4]. (The equivariance property of the moment map allows to prove that $\lambda(k \cdot x; s) = \lambda(x; \text{Ad}(k)(s))$ for any $k \in K$, which clearly implies that analytic stability is a property of K -orbits, but there is no obvious action of G on \mathfrak{k} extending the adjoint action of K giving a similar G -equivariance property of the maximal weights.)

It is easy to deduce from the results in [M] that if $\lambda(x; s) < 0$ for some s then $G \cdot x \cap \mu^{-1}(0) = \emptyset$. Hence if $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$ then $\lambda(x; s) \geq 0$ for any s . But this is not a

sufficient condition: to decide whether $G \cdot x \cap \mu^{-1}(0)$ is nonempty one needs to control in some way which of the maximal weights vanish.

A well known and elementary example which illustrates these phenomena is given by the action of the isometries of S^2 on tuples of points. Here S^2 denotes the unit sphere in \mathbb{R}^3 centered at 0 and endowed with the round metric. Let $K = \mathrm{SO}(3, \mathbb{R})$ be the group of orientation preserving isometries of S^2 . The complexification of K is $G = \mathrm{PSL}(2, \mathbb{C})$, which can be identified with the holomorphic automorphisms of $S^2 \simeq \mathbb{CP}^1$. Let $X = (S^2)^4$ and take on X the product Kaehler structure. The diagonal action of K on X clearly preserves the holomorphic structure and the symplectic form, and a moment map for it is given by sending any tuple $(x_1, \dots, x_4) \in S^2$ to its center of mass $\frac{1}{4}(x_1 + \dots + x_4)$ (here we identify $\mathbb{R}^3 \simeq \mathfrak{so}(3, \mathbb{R})^*$ using the vector product in \mathbb{R}^3). The reader can easily check that, if $\{x_1, x_2, x_3, x_4\} \subset S^2$ are distinct points, then

- $x = (x_1, x_2, x_3, x_4) \in X$ is analytically semistable,
- $x' = (x_1, x_1, x_2, x_3) \in X$ satisfies $G \cdot x' \cap \mu^{-1}(0) = \emptyset$ but all maximal weights $\lambda(x'; s)$ are nonnegative,
- $x'' = (x_1, x_1, x_2, x_2) \in X$ satisfies $G \cdot x'' \cap \mu^{-1}(0) \neq \emptyset$, but some of the maximal weights $\lambda(x; s)$ vanish.

(Of course, x' is semistable in the usual sense in GIT, and the closure of orbit $G \cdot x'$ contains $G \cdot x''$ and hence meets $\mu^{-1}(0)$.)

A complete solution to Question 1.1 was given by A. Teleman in [T]. However, the result in [T] has some limitations. First, it is assumed that X satisfies a condition called energy-completeness (see [T, Definition 3.8]). Second, when giving a sufficient condition for a point $x \in X$ to satisfy $G \cdot x \cap \mu^{-1}(0)$ it is assumed that the Lie algebra \mathfrak{g}_x of the stabilizer G_x is reductive (see [T, Definition 3.12]). This is a little bit unsatisfactory: it might be preferable to obtain the reductivity of \mathfrak{g}_x as a consequence of a simpler condition involving exclusively the maximal weights, not any information on the stabilizer of x .

In this paper we propose an alternative answer to Question 1.1 based on viewing the maximal weights as defining a function on the boundary $\partial_\infty(K \backslash G)$ of the symmetric space $K \backslash G$. Such boundary exists by the general theory of Hadamard spaces, of which $K \backslash G$ is an instance (see [B, E]), and it is homeomorphic to a sphere of dimension one unit less than that of $K \backslash G$. To prove our results we still require some technical restrictions to be satisfied by X ; namely, we assume that the moment map (resp. the vector fields generated by the infinitesimal action) grows quadratically (resp. linearly) with respect to the distance function from a given base point. More precisely: given a biinvariant metric on \mathfrak{k} we require that there exists a point $x_0 \in X$ and a constant C such that for any $x \in X$ and any $s \in \mathfrak{k}$ we have

$$(1.1) \quad |\xi_s(x)| \leq C |s| (1 + d_X(x, x_0))$$

$$(1.2) \quad |\mu(x)| \leq C (1 + d_X(x, x_0)^2)$$

where $\xi_s \in C^\infty(TX)$ is the vector field generated by the infinitesimal action of s and d_X is the distance function between points in X . These conditions are satisfied e.g. when

X is compact or when X is a vector space and the action of K is linear. (On the other hand, in this paper we do not assume any completeness condition as in [T].)

Assuming (1.1) and (1.2) we construct in Section 3.3 the maximal weight function

$$\lambda_x : \partial_\infty(K \backslash G) \rightarrow \mathbb{R} \cup \{\infty\}$$

for any $x \in X$ as an appropriate limit of a normalization of the integral of the moment map

$$\psi_x : K \backslash G \rightarrow \mathbb{R}.$$

The integral of the moment map was defined in [M] as a function $G \rightarrow \mathbb{R}$, and it was observed in [M, Proposition 3.4] that it is invariant under the action of K on the left on G . The boundary $\partial_\infty(K \backslash G)$ carries an action of G extending the right action on $K \backslash G$ by isometries, and we prove in Lemma 3.4 that for any $x \in X$ and $g \in G$ we have

$$(1.3) \quad \lambda_{g \cdot x}(y) = \lambda_x(y \cdot g).$$

This property is a consequence of the cocycle property satisfied by the integral of the moment map (see formula (3.6) below).

We say that $x \in X$ is analytically stable if for any $y \in \partial_\infty(K \backslash G)$ we have $\lambda_x(y) > 0$. A point $x \in X$ is said to be analytically polystable if for any $y \in \partial_\infty(K \backslash G)$ we have $\lambda_x(y) \geq 0$ and for any $y \in \partial_\infty(K \backslash G)$ such that $\lambda_x(y) = 0$ there exists some $y' \in \partial_\infty(K \backslash G)$ such that $\lambda_x(y') = 0$ and the points y, y' can be connected by a geodesic in $K \backslash G$.

The next theorem is the main result of the paper. It will be proved in Section 4.

Theorem 1.2. *Let $x \in X$ be any point and let $G_x = \{g \in G \mid g \cdot x = x\}$ be its stabilizer.*

- (1) *If x is analytically stable (resp. analytically polystable) then $g \cdot x$ is analytically stable (resp. analytically polystable) for each $g \in G$.*
- (2) *The intersection $G \cdot x \cap \mu^{-1}(0)$ consists of at most one K -orbit.*
- (3) *x is analytically stable if and only if $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$ and G_x is finite.*
- (4) *x is analytically polystable if and only if $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$. If this is the case, then G_x is reductive.*

Note that statement (1) follows immediately from the equivariance property (1.3). (2) and (3) are well known, but we also prove them for completeness (the proof we give of (3), using the index of maps between spheres, is new to the best of our knowledge).

For any $s \in \mathfrak{k}$ of unit length, let e_s be the boundary point of $K \backslash G$ to which the geodesic $t \mapsto [e^{its}]$ converges as $t \rightarrow \infty$. We have: $\lambda(x; s) = \lambda_x(e_s)$ for any $s \in \mathfrak{k}$ of unit norm, $\lambda(x; \alpha s) = \alpha \lambda(x; s)$ for any positive real number α , and any point in $\partial_\infty(K \backslash G)$ is of the form e_s for some s . Hence, the notion of analytic stability in the present paper coincides with the notion given in [M]. In order to rephrase the definition of polystability in terms of the functions $\lambda(x; s)$ we introduce the following definitions. Two skew Hermitian endomorphisms a, b of a complex vector space V are said to be opposed if $\mathbf{i}a$ and $-\mathbf{i}b$ have the same spectrum, say $\{\lambda_0 < \dots < \lambda_r\} \subset \mathbb{R}$, and the growing filtrations W_a^\bullet and

W_b^\bullet defined as

$$W_a^j = \bigoplus_{i \leq j} \text{Ker}(\mathbf{i}a - \lambda_i \text{Id}), \quad W_b^j = \bigoplus_{i \geq r-j} \text{Ker}(\mathbf{i}b + \lambda_i \text{Id})$$

satisfy

$$V = \bigoplus_{p+q=r} W_a^p \cap W_b^q.$$

Two elements $u, v \in \mathfrak{k}$ are said to be opposed if u and $-v$ belong to the same adjoint orbit in \mathfrak{k} and $\text{ad}(u), \text{ad}(v)$ are opposed endomorphisms of \mathfrak{g} . For example, for any $u \in \mathfrak{k}$ the elements $u, -u$ are opposed. Note also that if u has unit norm and $-v$ belongs to the adjoint orbit through u , then v also has unit norm because the norm in \mathfrak{k} is biinvariant. The following lemma is a consequence of Lemma 2.3 below.

Lemma 1.3. *A point $x \in X$ is polystable if $\lambda(x; s) \geq 0$ for any $s \in \mathfrak{k}$ and if, for any nonzero $s \in \mathfrak{k}$ such that $\lambda(x; s) = 0$, there exists some $u \in \mathfrak{k}$ which is opposed to s and such that $\lambda(x; u) = 0$.*

The results in this paper can be seen as an analytic version of part of the results in Chapter 2 of [MFK]. Mumford’s point of view is that the maximal weights in the case of projective varieties define a function on the flag complex $\Delta(G)$, which is the set of rational points at infinity of G and can be naturally thought as an algebraic version of the boundary $\partial_\infty(K \backslash G)$. More precisely, the function $\Delta(G) \ni \delta \mapsto \nu^L(x, \delta)$ defined in p. 59 of [op. cit.] is the analogue of our function λ_x . When X is projective and its symplectic structure is the restriction of the Fubini–Study structure on the projective space, statement (3) in Theorem 1.2, combined with Kempf–Ness’s results (see [Sch] for an excellent survey), implies the usual Hilbert–Mumford criterion for stability in GIT, and this explains the title of the present paper. On the other hand, statement (4) in Theorem 1.2 gives a characterization of which points x in a linear representation of a reductive group G have closed orbit $G \cdot x$ in terms uniquely of maximal weights, and this seems to be a new result (note that antipodal points in $\Delta(G)$, as defined in Definition 2.8, p. 61 in [op. cit.], correspond to pairs of points in $\partial_\infty(K \backslash G)$ which can be connected by a geodesic).

The rest of the paper is organized as follows. In Section 2 we recall the definition and some basic facts on the boundary at infinity of the symmetric space $K \backslash G$. In Section 3 we construct the maximal weight functions $\lambda_x : \partial_\infty(K \backslash G) \rightarrow \mathbb{R} \cup \{\infty\}$. In Section 4 we give the proof of Theorem 1.2 and, finally, in Section 5 we prove Lemma 2.3.

2. THE SYMMETRIC SPACE $K \backslash G$ AND ITS BOUNDARY AT INFINITY $\partial_\infty(K \backslash G)$

2.1. The boundary $\partial_\infty(K \backslash G)$. The coset space $K \backslash G$ has a natural structure of differentiable manifold. We consider on it the action of G given by multiplication on the right: $[g] \cdot h = [gh]$ for any $g, h \in G$. Let $x_0 \in K \backslash G$ denote the class of the identity element $1_G \in G$. Choose a biinvariant Euclidean norm on \mathfrak{k} . This induces a unique G -invariant Riemannian metric on $K \backslash G$, because the action $T(K \backslash G) \circ G$ given by differentiating

right multiplication is transitive, the stabilizer of the fiber $T_{x_0}(K \backslash G)$ over the identity element $1_G \in G$ is K , and the action of K on $T_{x_0}(K \backslash G)$ can be identified with the adjoint action of K on \mathfrak{k} (via the natural identification $T_{x_0}(K \backslash G) \simeq \mathfrak{k}$). The geodesics corresponding to this metric are given by maps $t \mapsto [e^{its}g] \in K \backslash G$ for any $s \in \mathfrak{k}$ and $g \in G$.

The invariant metric on $K \backslash G$ has nonpositive curvature (see [E]) so, endowed with it, $K \backslash G$ is a Hadamard space. So the general theory of Hadamard spaces (see for example [B]) implies that there is a naturally defined boundary at infinity $\partial_\infty(K \backslash G)$. This can be described in concrete terms using geodesic rays i.e. maps

$$\gamma : (0, \infty) \rightarrow K \backslash G$$

giving a parametrization by arc of a portion of geodesic. Let d denote the distance function between points in $K \backslash G$. Two geodesic rays γ_0, γ_1 are declared to be equivalent $\gamma_0 \sim \gamma_1$ if the distance $d(\gamma_0(t), \gamma_1(t))$ is bounded independently of t . This is an equivalence relation on the set of geodesic rays, and the boundary at infinity of $K \backslash G$ is the set of equivalence classes:

$$\partial_\infty(K \backslash G) = \{ \text{geodesic rays} \} / \sim .$$

If $\gamma : (0, \infty) \rightarrow K \backslash G$ is a geodesic ray and $g \in G$ then we define $\gamma \cdot g$ to be the geodesic ray whose value at t is $\gamma(t) \cdot g$. This defines a right action of G on the set of geodesic rays. Since the action of G on the right on $K \backslash G$ is by isometries, this action on the set of geodesic rays preserves the equivalence \sim and hence descends to an action on $\partial_\infty(K \backslash G)$.

Let $S(\mathfrak{k}) \subset \mathfrak{k}$ denote the unit sphere. For any $s \in S(\mathfrak{k})$ we define $e_s \in \partial_\infty(K \backslash G)$ to be the class of the geodesic ray $\eta_s : (0, \infty) \rightarrow K \backslash G$ defined as $\eta_s(t) = [e^{its}]$. Then the map $e : S(\mathfrak{k}) \ni s \mapsto [e_s] \in \partial_\infty(K \backslash G)$ is a bijection (see Section II.2 in [B]). We endow $\partial_\infty(K \backslash G)$ with the topology which makes e a homeomorphism. Then the action of G on $\partial_\infty(K \backslash G)$ is by homeomorphisms. For each $s \in S(\mathfrak{k})$ and any $g \in G$ define $s \cdot g \in S(\mathfrak{k})$ by the property that

$$e_s \cdot g = e_{s \cdot g}.$$

We remark that the boundary $\partial_\infty(K \backslash G)$ is independent of the chosen biinvariant metric on \mathfrak{k} . Indeed, geodesic rays do not depend on the choice of metric (they are always of the form $t \mapsto [e^{ist}g]$) and neither does the equivalence relation \sim on geodesic rays, because the distance functions on $K \backslash G$ induced by two choices of biinvariant metric on \mathfrak{k} are uniformly comparable.

2.2. The case $K = \mathrm{U}(n)$ and $G = \mathrm{GL}(n, \mathbb{C})$. When $K = \mathrm{U}(n)$ (so that $G = \mathrm{GL}(n, \mathbb{C})$) the action of G on $S(\mathfrak{u}(n))$ can be computed in concrete terms, as we will shortly see. Define the logarithm map $\log : G \rightarrow \mathfrak{k}$ by the condition that $\log(g) = u$ if $g = ke^{iu}$ is the Cartan decomposition of g , so that $k \in K$ and $u \in \mathfrak{k}$. Let $s \in S(\mathfrak{u}(n))$. The matrix $\mathbf{i}s$ is Hermitian symmetric, so it diagonalizes and has real eigenvalues, say $\lambda_1 < \dots < \lambda_r$. Let $V_j = \mathrm{Ker}(\lambda_j - \mathbf{i}s)$ be the eigenspace corresponding to λ_j and define $V^k = V_1 \oplus \dots \oplus V_k$ for any integer $k \geq 1$. Take any $g \in G$ and define

$$V_j^\infty = (g^{-1}(V_{j-1}))^\perp \cap g^{-1}(V_j),$$

where V^\perp denotes the orthogonal of V . Then we have a direct sum decomposition $\mathbb{C}^n = \bigoplus V_j^\infty$. Define $\rho_g(s) \in \mathfrak{u}(n)$ by the conditions that $\rho_g(s)$ preserves each V_j^∞ and that the restriction of $\rho_g(s)$ to V_j^∞ is given by multiplication by $-\mathbf{i}\lambda_j$. We claim that $\rho_g(s)$ is equal to $s \cdot g$. This is equivalent to the statement

$$(2.4) \quad \rho_g(s) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log(e^{\mathbf{i}\tau s} g).$$

To prove (2.4) one can argue as follows. Take a very small $\epsilon > 0$ (in particular, smaller than $\inf\{\lambda_j - \lambda_{j-1}\}/3$). Using the variational description of eigenvalues and eigenspaces of $\log(h)$, one proves that for big enough τ the eigenvalues of $s_\tau := \tau^{-1} \log(e^{\mathbf{i}\tau s} g)$ are contained in $\bigcup [\lambda_j - \epsilon, \lambda_j + \epsilon]$, and the number of eigenvalues in $[\lambda_j - \epsilon, \lambda_j + \epsilon]$ is equal to $\dim V_j$. Let V_j^τ be the direct sum of the eigenspaces of s_τ with eigenvalue contained in $[\lambda_j - \epsilon, \lambda_j + \epsilon]$. Then V_j^τ converges to V_j^∞ in the Grassmannian variety. Details are left to the reader.

Using the previous computations, we can also check that the map $e : S(\mathfrak{k}) \rightarrow \partial_\infty(K \setminus G)$ is a bijection. This is equivalent to proving the existence of a bound, for each $g \in G$ and $s \in S(\mathfrak{k})$, of the form $d([e^{\mathbf{i}t(s \cdot g)}], [e^{\mathbf{i}ts} g]) \leq C$, where the constant $C > 0$ is independent of t . Indeed, this implies that the geodesic ray $t \mapsto [e^{\mathbf{i}ts} g]$ is equivalent to $t \mapsto [e^{\mathbf{i}t(s \cdot g)}]$, which is $e_{s \cdot g}$. Details are left as an exercise to the reader.

2.3. Tori generated by elements in \mathfrak{k} . For any $s \in \mathfrak{k}$ we define the torus

$$T_s = \overline{\{\exp(ts) \mid t \in \mathbb{R}\}} \subset K.$$

Lemma 2.1. *For any $s \in \mathfrak{k}$ and any $g \in G$ we have $\dim T_s = \dim T_{s \cdot g}$.*

Proof. By Peter–Weyl theorem one can pick an embedding of Lie groups $K \hookrightarrow \mathrm{U}(n)$ which complexifying induces an inclusion $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$. Since the boundary at infinity does not depend on the choice of biinvariant metric, this inclusion induces an inclusion of boundaries $\partial_\infty(K \setminus G) \hookrightarrow \partial_\infty(\mathrm{U}(n) \setminus \mathrm{GL}(n, \mathbb{C}))$, which is equivariant with respect to the natural action of G on $\partial_\infty(K \setminus G)$ and the action of G on $\partial_\infty(\mathrm{U}(n) \setminus \mathrm{GL}(n, \mathbb{C}))$ given by the inclusion $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ (see the proof of Lemma 5.4 for details). All this implies that it suffices to consider the case $K = \mathrm{U}(n)$. But if $s \in \mathfrak{u}(n)$ then the dimension of T_s depends uniquely on the eigenvalues of s (namely, $\dim T_s$ is equal to the dimension of the \mathbb{Q} -vector space spanned by the eigenvalues of s). On the other hand, the observations in Section 2.2 imply that for any $s \in \mathfrak{u}(n)$ and $g \in \mathrm{GL}(n, \mathbb{C})$ the endomorphisms $s, s \cdot g \in \mathrm{End} \mathbb{C}^n$ have the same set of eigenvalues, so we certainly have $\dim T_s = \dim T_{s \cdot g}$. \square

2.4. Geodesically connected points. Two points in $\partial_\infty(K \setminus G)$ are said to be geodesically connected if there is a geodesic in $K \setminus G$ which converges on one side to one of the points and on the other side to the other point. This definition is independent of the biinvariant metric on \mathfrak{k} because the set of geodesics in $K \setminus G$ and the notion of convergence of rays to points in $\partial_\infty(K \setminus G)$ do not depend on the metric on \mathfrak{k} . A trivial example:

Example 2.2. *For any $s \in S(\mathfrak{k})$ the points $e_s, e_{-s} \in \partial_\infty(K \setminus G)$ are geodesically connected by the geodesic $t \mapsto [e^{\mathbf{i}ts}]$.*

A concrete translation into algebraic terms of the condition of being geodesically connected can be given using the notion of opposed elements in \mathfrak{k} defined in the Introduction:

Lemma 2.3. *Given $u, v \in S(\mathfrak{k})$, the points $e_u, e_v \in \partial_\infty(K \backslash G)$ are geodesically connected if and only if u, v are opposed.*

To avoid an excessive detour from our arguments we postpone the proof of the lemma to Section 5 at the end of the paper.

A more synthetic characterization of geodesic connectedness may be given in terms of parabolic subgroups. We state such translation for the sake of completeness, but we will not use it in the sequel. Recall that a parabolic subgroup of G is by definition the stabilizer of a point in $\partial_\infty(K \backslash G)$. It is almost a tautology that the stabilizer of $e_s \in \partial_\infty(K \backslash G)$ is the subgroup $P_s \subset G$ consisting of all $g \in G$ such that $e^{its}ge^{-its}$ stays bounded as $t \rightarrow \infty$, so that all parabolic subgroups of G are of the form P_s for some $s \in S(\mathfrak{k})$. The maximal reductive subgroups of P_s are called the Levi subgroups (they are all pairwise conjugate). Two parabolic subgroups $P_s, P_{s'} \subset G$ are said to be opposed if $P_s \cap P_{s'}$ is a Levi subgroup both of P_s and $P_{s'}$. Now, e_s and $e_{s'}$ are geodesically connected if and only if P_s and $P_{s'}$ are opposed and $s, -s'$ belong to the same coadjoint orbit in \mathfrak{k} .

3. MAXIMAL WEIGHTS AS A MAP $\lambda_x : \partial_\infty(K \backslash G) \rightarrow \mathbb{R} \cup \{\infty\}$

We now come back to the situation considered in the Introduction, so that $K \curvearrowright X$ is a Hamiltonian action of a compact Lie group K on a Kaehler manifold X preserving the complex structure, and we consider the extension of this action to a holomorphic action $G \curvearrowright X$ of the complexification $G = K^\mathbb{C}$.

3.1. The integral of the moment map. Denote by $\pi : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{ik} \rightarrow \mathfrak{ik}$ the projection to the second factor. Let $r_{g^{-1}} : G \rightarrow G$ be the map given by multiplication by g^{-1} on the right, and let $Dr_{g^{-1}} : T_g G \rightarrow T_{1_G} G \simeq \mathfrak{g}$ be its derivative. For any $v \in T_g G$ we define $v \cdot g^{-1} := Dr_{g^{-1}}(v) \in \mathfrak{g}$.

For any $x \in X$ we define a one form $\sigma_x \in \Omega^1(G)$ as follows:

$$\sigma_x(g)(v) := \langle \mu(g \cdot x), -\mathfrak{i}\pi(v \cdot g^{-1}) \rangle$$

for any $v \in T_v G$. It is immediate to deduce from the definition that for any $g, h \in G$ and any $v \in T_g G$ we have $\sigma_x(gh)(v \cdot h) = \sigma_x(g)(v)$, so that

$$(3.5) \quad \sigma_{hx} = r_h^* \sigma_x.$$

By [M, Lemma 3.1] the form σ_x is exact. Hence we may define $\Psi_x : G \rightarrow \mathbb{R}$ to be the unique function such that $\Psi_x(1_G) = 0$ and $d\Psi_x = \sigma_x$. We call Ψ_x the **integral of the moment map**. Property (3.5) implies the following cocycle formula:

$$(3.6) \quad \Psi_x(g) + \Psi_{g \cdot x}(h) = \Psi_x(hg)$$

for any $x \in X$ and $g, h \in G$.

3.2. Asymptotics of the integral of the moment map. Given $s \in \mathfrak{k}$ we define $\mu_s(x) = \langle \mu(x), s \rangle$ for any $x \in X$ and for any $t \in \mathbb{R}$ we define $\lambda_t(x, s) = \mu_s(e^{its} \cdot x)$. For any $s \in \mathfrak{g}$ we denote by

$$\xi_s \in \mathcal{C}^\infty(TX)$$

the vector field generated by the infinitesimal action of s . Since the action of G on X is holomorphic we have $\xi_{is} = I\xi_s$, where $I \in \mathcal{C}^\infty(\text{End } TX)$ is the complex structure on X . Using the defining properties of the moment map we compute:

$$\begin{aligned} \partial_t \lambda_t(x; s) &= \partial_t \langle \mu(e^{its} \cdot x), s \rangle = \omega(\xi_s, I\xi_s)(e^{its} \cdot x) \\ (3.7) \quad &= \langle \xi_s(e^{its} \cdot x), \xi_u(e^{its} \cdot x) \rangle = |I\xi_s(e^{its} \cdot x)|^2, \end{aligned}$$

where ∂_t denotes the derivative with respect to t . This implies

$$(3.8) \quad \lambda_t(x; s) = \langle \mu(x), s \rangle + \int_0^t |\xi_s(e^{i\tau s} \cdot x)|^2 d\tau,$$

and in particular $\lambda_t(x; s)$ is nondecreasing as a function of t .

It follows from the definition of σ_x that for any $s \in \mathfrak{k}$

$$\Psi_x(e^{its}) = \int_0^t \lambda_\tau(x; s) d\tau.$$

Since λ_τ is nondecreasing we deduce that

$$(3.9) \quad \lim_{t \rightarrow \infty} \frac{\Psi_x(e^{its})}{t} = \lambda(x; s) := \lim_{t \rightarrow \infty} \lambda_t(x; s) \in \mathbb{R} \cup \{\infty\}.$$

The limit $\lambda(x; s)$ is what was defined to be the maximal weight in [M]. When it is necessary to be more specific, we will say that $\lambda(x; s)$ is the maximal weight of the action of K on X and we will denote it by $\lambda^K(x; s)$ (this will be the case in Section 4, where different symmetry groups will be considered simultaneously).

We end this section by showing how the growth of the integral of the moment map can be used to bound the distance between points in X . Recall that d_X denotes the distance function between pairs of points in X .

Lemma 3.1. *Let $g \in G$ and $s \in \mathfrak{k}$. If $\Psi_x(e^{its}g)t^{-1}$ is bounded uniformly on t , then $d_X(e^{its}g \cdot x, x)t^{-1/2}$ converges to 0 as $t \rightarrow \infty$.*

Proof. Using (3.6) and (3.9) we compute

$$\lim_{t \rightarrow \infty} \frac{\Psi_x(e^{its}g)}{t} = \lim_{t \rightarrow \infty} \frac{\Psi_x(g) + \Psi_{g \cdot x}(e^{its})}{t} = \lim_{t \rightarrow \infty} \frac{\Psi_{g \cdot x}(e^{its})}{t} = \lambda(g \cdot x; s).$$

So, if $\Psi_x(e^{its}g)t^{-1}$ is bounded uniformly on t , then, by (3.8), $\int_0^\infty |\xi_s(e^{i\tau s}g \cdot x)|^2 d\tau < \infty$. Since, on the other hand,

$$d_X(e^{its}g \cdot x, g \cdot x) \leq \int_0^t |\xi_{is}(e^{i\tau s}g \cdot x)| d\tau = \int_0^t |\xi_s(e^{i\tau s}g \cdot x)| d\tau,$$

the following lemma applied to $f(\tau) = |\xi_s(e^{i\tau s}g \cdot x)|$ implies that $d_X(e^{its}g \cdot x, g \cdot x)t^{-1/2}$ converges to 0 as $t \rightarrow \infty$. The lemma is finished by applying the triangular inequality. \square

Lemma 3.2. *Let $f : (0, \infty) \rightarrow \mathbb{R}_{\geq 0}$ be a nonnegative square integrable function, so that we have $\int_0^\infty f^2(\tau) d\tau < \infty$. Then*

$$\left(\int_0^t f(\tau) d\tau \right) t^{-1/2} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. Let $E = \int_0^\infty f^2(\tau) d\tau$, let $\epsilon > 0$ be any real number and choose $t_0 > 0$ in such a way that $\int_0^{t_0} f(\tau)^2 d\tau \geq (1 - \epsilon)E$, so that for any $t \geq t_0$ we have

$$\int_{t_0}^t f(\tau)^2 d\tau \leq \epsilon E.$$

Then we compute, using Cauchy–Schwartz and the previous estimate:

$$\begin{aligned} \int_0^t f(\tau) d\tau &= \int_0^{t_0} f(\tau) d\tau + \int_{t_0}^t f(\tau) d\tau = \int_0^{t_0} f(\tau) d\tau + \int_{t_0}^t (\epsilon^{-1/2} f(\tau)) \epsilon^{1/2} d\tau \\ &\leq (Et_0)^{1/2} + (E\epsilon(t - t_0))^{1/2} = E^{1/2}(t_0^{1/2} + \epsilon^{1/2}(t - t_0)^{1/2}). \end{aligned}$$

The result follows by observing that $(t_0^{1/2} + \epsilon^{1/2}(t - t_0)^{1/2})t^{-1/2} \rightarrow \epsilon^{1/2}$ as $t \rightarrow \infty$. \square

3.3. Extending the integral of the moment map to $\partial_\infty(K \backslash G)$. The cocycle condition (3.6) and the fact that for any $y \in X$ the restriction of Ψ_y to $K \subset G$ vanishes identically (which follows immediately from the definition) implies that $\Psi_x(kg) = \Psi_x(g)$ for each $k \in K$ and $g \in G$, so that Ψ_x descends to a map

$$\psi_x : K \backslash G \rightarrow \mathbb{R}.$$

Recall that d denotes the distance function between pairs of points in $K \backslash G$, and that $x_0 \in K \backslash G$ denotes the class of the identity element in G . We are next going to prove that the function $\phi_x : K \backslash G \rightarrow \mathbb{R}$ defined as

$$\phi_x(z) = \frac{\psi_x(z)}{d(z, x_0)}$$

extends to a function on the boundary $\partial_\infty(K \backslash G)$. For any geodesic ray $\gamma : [0, \infty) \rightarrow K \backslash G$ we define

$$\lambda_x(\gamma) = \lim_{t \rightarrow \infty} \phi_x(\gamma(t)) \in \mathbb{R} \cup \{\infty\}.$$

The extendability of ϕ_x is equivalent to the following lemma.

Proposition 3.3. *If the geodesic rays γ_0, γ_1 satisfy $\gamma_0 \sim \gamma_1$, then*

$$\lambda_x(\gamma_0) = \lambda_x(\gamma_1).$$

Proof. We may write $\gamma_j(t) = [e^{its_j} g_j]$ for $j = 0, 1$, where $s_j \in \mathfrak{k}$ and $g_j \in G$. Assume that $\lambda_x(\gamma_0)$ is finite. Then Lemma 3.1 implies that $d(e^{its_0} g_0 \cdot x, x_0) t^{-1/2}$ converges to 0 as $t \rightarrow \infty$. Since $\gamma_0 \sim \gamma_1$, we may bound $d(\gamma_0(t), \gamma_1(t)) \leq \kappa$ uniformly on t . It follows

that, for any t , we may take a smooth function $\rho_t : [0, 1] \rightarrow G$ such that $\rho_t(j) = e^{is_j t} g_j$ for $j = 0, 1$, and such that

$$(3.10) \quad \int_0^1 |\partial_\nu \rho_t(\nu) \cdot \rho_t(\nu)^{-1}| d\nu \leq \kappa.$$

This bound, together with $d(e^{its_0} g_0 \cdot x, x_0) t^{-1/2} \rightarrow 0$ and assumption (1.1), implies that, for any ν , $d(\rho_t(\nu) \cdot x, x_0) t^{-1/2} \rightarrow 0$. Using assumption (1.2) we get $|\mu(\rho_t(\nu) \cdot x)| t^{-1} \rightarrow 0$. Since $d\Psi_x = \sigma_x$, the previous formula together with (3.10) implies that

$$\lim_{t \rightarrow \infty} \frac{|\Psi_x(e^{its_0} g_0) - \Psi_x(e^{its_1} g_1)|}{t} = 0,$$

from which we deduce $\lambda_x(\gamma_0) = \lambda_x(\gamma_1)$. This immediately implies, arguing by contradiction, that if $\lambda_x(\gamma_0) = \infty$ then $\lambda_x(\gamma_1) = \infty$. For if $\lambda_x(\gamma_1) < \infty$ then, reversing the roles of γ_0 and γ_1 in the previous arguments, we deduce that $\lambda_x(\gamma_0) = \lambda_x(\gamma_1) < \infty$. \square

Using the previous lemma we may define $\lambda_x(y) := \lambda_x(\gamma) \in \mathbb{R} \cup \{\infty\}$ for any $y \in \partial_\infty(K \setminus G)$, where γ is any geodesic ray representing y . In this way we obtain a well defined map

$$\lambda_x : \partial_\infty(K \setminus G) \rightarrow \mathbb{R} \cup \{\infty\},$$

which we call the maximal weight function. We now prove a crucial equivariance property of the maximal weights.

Lemma 3.4. *For any $y \in \partial_\infty(K \setminus G)$ and any $g \in G$ we have $\lambda_{g \cdot x}(y) = \lambda_x(y \cdot g)$.*

Proof. Let γ be a geodesic ray representing y . Using the cocycle formula (3.6) we compute

$$\begin{aligned} \lambda_{g \cdot x}(y) &= \lim_{t \rightarrow \infty} \phi_{g \cdot x}(\gamma(t)) = \lim_{t \rightarrow \infty} \frac{\Psi_{g \cdot x}(\gamma(t))}{d(\gamma(t), x_0)} = \lim_{t \rightarrow \infty} \frac{\Psi_x(\gamma(t)g) - \Psi_x(g)}{d(\gamma(t), x_0)} \\ &= \lim_{t \rightarrow \infty} \frac{\Psi_x(\gamma(t)g)}{d(\gamma(t), x_0)} = \lim_{t \rightarrow \infty} \frac{\Psi_x(\gamma(t)g)}{d(\gamma(t)g, x_0)} = \lambda_x(y \cdot g), \end{aligned}$$

since $t \mapsto \gamma(t)g$ represents $y \cdot g$ and the quotient $d(\gamma(t), x_0)/d(\gamma(t)g, x_0)$ converges to 1 as $t \rightarrow \infty$, because $d(\gamma(t), x_0) = d(\gamma(t)g, x_0g)$ converges to ∞ and by the triangular inequality $|d(\gamma(t)g, x_0g) - d(\gamma(t)g, x_0)| \leq d(x_0g, x_0)$, which is independent of t . \square

3.4. Some easy properties of maximal weights. In the next two lemmata x, x' denote points in X and s, s' denote elements in \mathfrak{k} . Recall that ξ_s denotes the vector field on X generated by the infinitesimal action of s .

Lemma 3.5. *If $\lambda(x; s) = \lambda(x; -s) = 0$ then $\xi_s(x) = 0$.*

Proof. By (3.8) $\lambda(x; s) = 0$ implies that $\langle \mu(x), s \rangle \leq 0$, and $\lambda(x; -s) = 0$ implies $\langle \mu(x), -s \rangle \leq 0$. Combining both inequalities we have $\langle \mu(x), s \rangle = 0$. Using again (3.8) and the equality $\lambda(x; s) = \langle \mu(x), s \rangle$ we obtain $\xi_s(x) = 0$. \square

Lemma 3.6. *If $[s, s'] = 0$ then $\lambda(x; s + s') = \lambda(x; s) + \lambda(x'; s)$.*

Proof. We have

$$\begin{aligned}
\lambda(x; s + s') &= \lim_{t \rightarrow \infty} \langle \mu(e^{it(s+s')} \cdot x), s + s' \rangle \\
&= \lim_{t \rightarrow \infty} \langle \mu(e^{its} e^{its'} \cdot x), s \rangle + \langle \mu(e^{its} e^{its'} \cdot x), s' \rangle \quad \text{by linearity and } [s, s'] = 0 \\
&= \lim_{t \rightarrow \infty} \langle \mu(e^{its} \cdot x), s \rangle + \langle \mu(e^{its'} \cdot x), s' \rangle \quad \text{by equivariance of } \mu \text{ and } [s, s'] = 0 \\
&= \lambda(x; s) + \lambda(x; s').
\end{aligned}$$

□

4. PROOF OF THEOREM 1.2

4.1. Proofs of (1) and (2). Statement (1) follows immediately from Lemma 3.4, and the fact that the action of G on $\partial_\infty(K \setminus G)$ extends the action by isometries on $K \setminus G$ (so that the action on $\partial_\infty(K \setminus G)$ of any element in G sends geodesically connected points to geodesically connected points). (2) is well known, but we recall the argument for the sake of completeness. If $G \cdot x \cap \mu^{-1}(0)$ contains two different K -orbits, say $\mathcal{O}_1, \mathcal{O}_2 \subset K$, then by Cartan's decomposition we may find points $x_j \in \mathcal{O}_j$ such that $x_2 = e^{its} \cdot x_1$. We then have $\mu_s(x_2) = \mu_s(e^{is} \cdot x) = \mu_s(x_1) = 0$. By (3.7) we have

$$\mu_s(e^{is} \cdot x) - \mu_s(x) = \int_0^1 |\xi_s(e^{i\tau s} \cdot x_1)|^2 d\tau,$$

which implies that $\xi_s(e^{i\tau s} \cdot x_1)$ vanishes for all $\tau \in [0, 1]$, so that the action of $\{e^{its} \mid t \in \mathbb{R}\}$ fixes x_1 . Consequently, $x_2 = x_1$, which implies $\mathcal{O}_1 = \mathcal{O}_2$, a contradiction.

4.2. Proof of (3). Assume that x is analytically stable, so that for any $s \in S(\mathfrak{k})$ we have $\lambda(x; s) > 0$. The usual argument to prove that $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$ is based on the identification between the zeroes of the moment map and the critical values of the integral of the moment map Ψ_x , and the fact that analytic stability implies that Ψ_x is proper. Instead, we give here a topological argument. The condition $\lambda(x; s) > 0$ implies that there is some τ_s such that if $t \geq \tau_s$ then $\langle \mu(e^{its} \cdot x), s \rangle > 0$. Since the latter function is continuous and $S(\mathfrak{k})$ is compact, we may take some τ working for any choice of s , namely, such that:

$$(4.11) \quad \text{for any } t \geq \tau \text{ and any } s \in S(\mathfrak{k}) \text{ we have } \langle \mu(e^{its} \cdot x), s \rangle > 0.$$

Denote by $\alpha : \mathfrak{k}^* \simeq \mathfrak{k}$ the isomorphism given by the pairing $\langle \cdot, \cdot \rangle$. Property (4.11) implies that the image of the map $f : S(\mathfrak{k}) \rightarrow \mathfrak{k}$ given $f(s) = \alpha \circ \mu(e^{i\tau s} \cdot x)$ is contained in $\mathfrak{k} \setminus \{0\}$, and furthermore there is a homotopy between f and the identity via maps from $S(\mathfrak{k})$ to $\mathfrak{k} \setminus \{0\}$. In other words, the index of f around $0 \in \mathfrak{k}$ is nontrivial, and this implies that there is some u inside the ball in \mathfrak{k} with boundary $S(\mathfrak{k})$ such that $f(u) = 0$, which is equivalent to $\mu(e^{i\tau u} \cdot x) = 0$. Now to prove that G_x is finite is equivalent to proving that G_y is finite, where $y = e^{i\tau u} \cdot x$. Since $\mu(y) = 0$ and $\lambda(y; s) > 0$ for any s , formula (3.8) implies that for any $s \in \mathfrak{k}$ the vector field ξ_s is nonzero at y . Consequently the stabilizer K_y is finite. Finally, the condition $\mu(y) = 0$ implies that G_y is the complexification of

K_y (this is proved by checking, using (3.7), that if ke^{iu} fixes y , $k \in K$ and $u \in \mathfrak{k}$, then $\xi_u(y) = 0$ and $k \in K_x$, see [S, Proposition 1.6]). Hence, G_y is also finite.

The converse implication in (3) is almost immediate: if $y \in G \cdot x \cap \mu^{-1}(0)$ and G_x is finite, then G_y is also finite. This implies that $\xi_u(y) \neq 0$ for any $u \in S(\mathfrak{k})$, and now (3.7) implies that $\lambda(y; s) > 0$, so y is analytically stable. From (1) it now follows that x is also analytically stable.

4.3. Proof of (4). We first prove that if x is polystable then $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$. Since (3) has been proved, we only need to consider strictly polystable points x (namely, unstable polystable points). So let $x \in X$ be such a point. Then one can choose $s \in S(\mathfrak{k})$ such that $\lambda_x(e_s) = 0$ and such that $\dim T_{s'} \leq \dim T_s$ for any other $s' \in S(\mathfrak{k})$ satisfying $\lambda_x(e_{s'}) = 0$.

Let $y = e_s$. Since x is polystable, there exists some $y' \in \partial_\infty(K \setminus G)$ which is geodesically connected to y and such that $\lambda_x(y') = 0$. Let $\gamma : \mathbb{R} \rightarrow K \setminus G$ be a parameterized geodesic in $K \setminus G$ connecting y and y' , and assume that $\gamma(t) = e^{iut}h$ for some $u \in \mathfrak{k}$ and $h \in G$. By (1) the point $w = h \cdot x \in X$ is polystable. If we set $u = s \cdot h^{-1}$ then we have $s' \cdot h^{-1} = -u$, since the points $e_{s \cdot h^{-1}}$ and $e_{s' \cdot h^{-1}}$ are connected by a geodesic passing through $x_0 \in K \setminus G$. By Lemma 3.4 we have $\lambda_w(e_u) = \lambda_w(e_{-u}) = 0$. In other words, $\lambda(w; u) = \lambda(w; -u) = 0$. Then Lemma 3.5 implies that $\xi_u(w) = 0$. Hence the group $\{\exp(ts) \mid t \in \mathbb{R}\} \subset K$ fixes w , and by continuity this implies that T_u fixes w . So, if \mathfrak{t}_u denotes the Lie algebra of T_u then for any $u' \in \mathfrak{t}_u$ we have $\xi_{u'}(w) = 0$.

Lemma 4.1. *For any $u' \in \mathfrak{t}_u$ we have $\lambda(w; u') = 0$.*

Proof. Since w is polystable we have $\lambda(w; u') \geq 0$ and $\lambda(w; -u') \geq 0$. Now, (3.8) together with $\xi_{u'}(w) = 0$ implies that $\lambda(w; u') = \langle \mu(w), u' \rangle = -\langle \mu(w), -u' \rangle = -\lambda(w; -u')$. \square

Lemmata 2.1 and 3.4 imply that u has the same maximality property as s , namely

$$(4.12) \quad \dim T_{u'} \leq \dim T_u \text{ for any } u' \in S(\mathfrak{k}) \text{ satisfying } \lambda_w(e_{u'}) = 0.$$

Let $K_u = \{k \in K \mid \text{Ad}(k)(u) = u\}$ be the centralizer of u . Then $T_u \subset K_u$ is obviously central. Let $K_0 = K_u/T_u$ and let \mathfrak{k}_0 be its Lie algebra. Consider the following maps:

- (1) the projection $\pi_u : \mathfrak{k}^* \rightarrow \mathfrak{k}_u^*$ induced by the inclusion $\mathfrak{k}_u \subset \mathfrak{k}$, and
- (2) the projection $\pi_0 : \mathfrak{k}_u^* \rightarrow \mathfrak{k}_0^*$ induced by any linear map $\mathfrak{k}_0 \rightarrow \mathfrak{k}_u$ which is a section of the projection $\mathfrak{k}_u \rightarrow \mathfrak{k}_u/\mathfrak{t}_u = \mathfrak{k}_0$ (π_0 is automatically a morphism of Lie algebras because \mathfrak{t}_u is central in \mathfrak{k}_u).

Let $X_u \subset X$ be the set of points fixed by all elements of T_u . Then X_u is a Kaehler submanifold of X and the group K_0 acts on it by isometries. A moment map for this action,

$$\mu^{K_0} : X_u \rightarrow \mathfrak{k}_0^*,$$

can be obtained by composing $\mu^{K_0} = \pi_0 \circ \pi_u \circ \mu|_{X_u}$.

We claim that $w \in X_u$ is stable with respect to the action of K_0 . First of all we observe that for any $u' \in \mathfrak{t}_u$ we have

$$\lambda^{K_0}(w; [u']) = \lambda^K(w; u'),$$

where on the left hand side we consider the maximal weights of the action of K_0 on X_u and $[u']$ denotes the class in $\mathfrak{k}_0 = \mathfrak{k}_u/\mathfrak{t}_u$ represented by u' , and on the right hand side we consider the weights of the action of K on X . It follows that

$$\lambda^{K_0}(w; [u']) \geq 0$$

for any $u' \in \mathfrak{k}_u$. We claim that the latter inequality is strict unless $[u'] = 0$. Indeed, if $[u'] \neq 0$ and $\lambda^{K_0}(w; [u']) = 0$ then, letting $T \subset K$ be the torus generated by T_u and by the closure of $\{\exp(tu') \mid t \in \mathbb{R}\}$, we would have, by Lemma 3.6 and arguing as in the proof of Lemma 4.1, $\lambda(w; v) = 0$ for any $v \in \mathfrak{t} = \text{Lie } T$. Choosing v in such a way that $\{\exp(tv) \mid t \in \mathbb{R}\}$ is dense in T we would furthermore have $\dim T_v > \dim T_u$, contradicting the maximality property (4.12).

Hence w is stable with respect to the action of K_0 on X_u , so by (3) there exists some $h \in G_0$ such that $\mu^{K_0}(g \cdot w) = 0$. This immediately implies that $\mu^{K_u}(g \cdot w) = 0$, where $\mu^{K_u} = \pi_u \circ \mu|_{X_u}$ is the moment map for the action of K_u on X_u (see Lemma 4.1). We now prove that we also have $\mu(g \cdot w) = 0$. Let us denote for convenience $z = g \cdot w$. Then z is fixed by the action of T_u . Take a decomposition

$$\mathfrak{k} = \mathfrak{k}_u^* \oplus \bigoplus_{\alpha} \mathfrak{k}_{\alpha}$$

in irreducible representations of T_u , so that \mathfrak{k}_u is the trivial representation and each \mathfrak{k}_{α} is nontrivial. This splitting induces a splitting of the dual vector space \mathfrak{k}^* , and we let $\mu(z) = \mu_u(z) + \sum \mu_{\alpha}(z)$ be the corresponding decomposition of μ . We clearly have $\mu_u(z) = \mu^{K_u}(z) = 0$. Now, since z is fixed by T_u , the equivariance of the moment map implies that each $\mu_{\alpha}(z)$ is a T_u invariant linear map $\mathfrak{k}_{\alpha} \rightarrow \mathbb{R}$. But each \mathfrak{k}_{α} is a nontrivial irreducible representation of T_u , so the following lemma implies that $\mu_{\alpha}(z) = 0$.

Lemma 4.2. *Let V be a finite dimensional vector space and let $\Gamma \curvearrowright V$ be an irreducible nontrivial linear action. Any Γ -invariant linear function $f : V \rightarrow \mathbb{R}$ vanishes identically.*

Proof. Take any nonzero $v \in V$ which is not fixed by Γ . Then the affine closure¹ $\langle \Gamma \cdot v \rangle_{\text{aff}}$ of $\Gamma \cdot v$ equal to V . Indeed, $\langle \Gamma \cdot v \rangle_{\text{aff}}$ is Γ -invariant and is not a point, so if it were a proper subspace of V then its translate containing the origin would be a proper nonzero invariant vector subspace of V , contradicting the irreducibility of $\Gamma \curvearrowright V$. Since f is Γ -invariant, f is constant on $\Gamma \cdot v$, and by linearity the restriction of f to $\langle \Gamma \cdot v \rangle_{\text{aff}}$ is also constant. Since $0 \in \langle \Gamma \cdot v \rangle_{\text{aff}}$ and f is linear, we must have $f = 0$. \square

Since $z \in G \cdot x$, we have proved that $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$.

The converse statement in (4) is almost immediate. Assume that $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$, and let $z \in G \cdot x \cap \mu^{-1}(0)$. By statement (1) it suffices to prove that z is polystable. Since $\mu(z) = 0$, (3.8) implies that $\lambda(z; s) \geq 0$ for any $s \in S(\mathfrak{k})$, and also $\lambda(z; s) = 0$ if and only if $\xi_s(z) = 0$. The latter implies that $\lambda(z; s) = 0$ if and only if $\lambda(z; -s) = 0$. Since e_s, e_{-s} are always geodesically connected (see Example 2.2), it follows that z is polystable.

¹If $X \subset V$, the affine closure $\langle X \rangle_{\text{aff}} \subset V$ is the set of finite sums $\sum \lambda_i x_i$ with $\sum \lambda_i = 1$ and $x_i \in X$.

It remains to prove that the stabilizer of polystable points is reductive. Since we have proved that if x is polystable then $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$, it suffices to prove that if $\mu(z) = 0$ then G_z is reductive. This follows from the well known observation that G_z is the complexification of the compact group $K_z = \{k \in K \mid k \cdot z = z\}$.

5. OPPOSED ELEMENTS IN \mathfrak{k} AND GEODESICALLY CONNECTED POINTS IN $\partial_\infty(K \backslash G)$

The main result of this section is the proof of Lemma 2.3, which will be given in the Section 5.3. In Sections 5.1 and 5.2 we state and prove some preliminary lemmata. Some of these results are probably well known to experts, but we prove them in some detail for the reader's convenience.

5.1. The K -orbits and the G -orbits in $\partial_\infty(K \backslash G)$ are the same. Recall that for any $s \in S(\mathfrak{k})$ the stabilizer of $e_s \in \partial_\infty(K \backslash G)$ is the parabolic subgroup

$$(5.13) \quad P_s = \{g \in G \mid e^{its} g e^{-its} \text{ stays bounded as } t \rightarrow \infty\}.$$

As previously, we denote by $x_0 \in K \backslash G$ the class of the identity element $1_G \in G$.

Lemma 5.1. *For any $s \in S(\mathfrak{k})$ the action of P_s on $K \backslash G$ (given by restricting the action of G) is transitive.*

Proof. Take any $s \in S(\mathfrak{k})$. Since $K \backslash G$ is connected, to prove the lemma it suffices to check that $x_0 \cdot P_s$ is open and closed in $K \backslash G$. With the aim of proving that $x_0 \cdot P_s$ is open, let us check that $\mathfrak{k} + \mathfrak{p}_s = \mathfrak{g}$, where $\mathfrak{p}_s \subset \mathfrak{g}$ is the Lie algebra of P_s . The endomorphism $\text{ad}(s) \in \text{End } \mathfrak{g}$ is semisimple because it preserves the extension to \mathfrak{g} of the biinvariant scalar product in \mathfrak{k} . Hence we may consider the decomposition in eigenspaces $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ of the action of $\text{ad}(s)$ on \mathfrak{g} , where each λ is real and $\text{ad}(s)$ acts on \mathfrak{g}_λ as multiplication by λ . It follows from (5.13) that $\mathfrak{p}_s = \bigoplus_{\lambda \leq 0} \mathfrak{g}_\lambda$. Let $c : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the conjugation map given by the identification $\mathfrak{g} \simeq \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, so that $\mathfrak{k} \subset \mathfrak{g}$ is the fixed point set of c . Since $c([a, b]) = [c(a), c(b)]$ for any $a, b \in \mathfrak{g}$, we have $c \circ \text{ad}(s) \circ c = -\text{ad}(s)$, which implies that c induces isomorphisms $\mathfrak{g}_\lambda \simeq \mathfrak{g}_{-\lambda}$ for each λ . Since \mathfrak{k} is the fixed point set of c , for any nonzero λ the intersection $\mathfrak{k} \cap (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})$ is equal to the graph of $c : \mathfrak{g}_{-\lambda} \rightarrow \mathfrak{g}_\lambda$. Combining this with the fact that $\mathfrak{p}_s = \bigoplus_{\lambda \leq 0} \mathfrak{g}_\lambda$, we deduce that $\mathfrak{k} + \mathfrak{p}_s = \mathfrak{g}$. This implies, by the inverse function theorem, that any $g \in G$ sufficiently near 1_G can be written as $g = k \cdot p$ for some $k \in K$ and $p \in P_s$, which means that $x_0 \cdot P_s$ contains a neighborhood of x_0 . Since P_s acts on $K \backslash G$ by homeomorphisms, this implies that $x_0 \cdot P_s$ contains a neighborhood of any of its points, so it is open.

Let $\alpha : P_s \rightarrow K \backslash G$ be the map $p \mapsto \alpha(p) = x_0 \cdot p$. Then α is the restriction of the quotient map $G \rightarrow K \backslash G$, which is proper because K is compact. Since $P_s \subset G$ is closed, it follows that α is also proper, so the intersection of $\alpha(P_s)$ with any compact subset of $K \backslash G$ is closed. Since $K \backslash G$ is locally compact, it follows that $x_0 \cdot P_s = \alpha(P_s)$ is closed. \square

Lemma 5.2. *If $y, y' \in \partial_\infty(K \backslash G)$ satisfy $y' = y \cdot g$ for some $g \in G$, then there exists some $k \in K$ such that $y' = y \cdot k$.*

Proof. Assume that $y' = y \cdot g = e_s$ and $z = x_0 \cdot g$. By Lemma 5.1 there exists some $p \in P_s$ such that $z \cdot p = x_0$. Hence $k := gp$ satisfies $y' = y \cdot k$ and $x_0 \cdot k = x_0$, which implies that $k \in K$. \square

The previous lemma implies that for any $s \in \mathfrak{k}$ of unit norm there is a right action of G on the adjoint orbit $\mathcal{O}_s = \text{Ad}(K) \cdot s \subset \mathfrak{k}$. Indeed, via the map $S(\mathfrak{k}) \ni u \mapsto e_u \in \partial_\infty(K \backslash G)$ the action of K on the boundary $\partial_\infty(K \backslash G)$ corresponds to the adjoint action on $S(\mathfrak{k})$. Since the K -orbits in $\partial_\infty(K \backslash G)$ are equal to the G -orbits, for any $s \in S(\mathfrak{k})$ we can identify \mathcal{O}_s with one of the G -orbits. And since the stabilizer of s is P_s , we obtain a natural identification $\mathcal{O}_s \simeq P_s \backslash G$.

5.2. The dense orbit of the action of P_{-s} on \mathcal{O}_s . Let $\mathcal{O}_s^* \subset \mathcal{O}_s$ denote the set of elements which are opposed to $-s$.

Lemma 5.3. *The set $\mathcal{O}_s^* \subset \mathcal{O}_s$ is open, dense, and connected.*

Proof. To prove the lemma we check that \mathcal{O}_s carries a structure of complex connected manifold with respect to which $\mathcal{O}_s \setminus \mathcal{O}_s^* \subset \mathcal{O}_s$ is an analytic subvariety of dimension $< \dim \mathcal{O}_s$. Since K is connected, $\mathcal{O}_s = \text{Ad}(K)(s)$ is also connected. Let $\mathcal{O}_{\text{ad}(s)} \subset \text{End } \mathfrak{g}$ be the adjoint orbit of $\text{ad}(s)$ under the action of the vector space automorphisms of \mathfrak{g} . Let $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ be the eigenspace decomposition of the action of $\mathbf{i} \text{ad}(s)$ on \mathfrak{g} , as in the proof of Lemma 5.1. Let \mathcal{F} be the set of growing filtrations $(W^\mu)_{\mu \in \mathbb{R}}$ of complex subspaces of \mathfrak{g} satisfying $\dim W^\mu = \sum_{\lambda \leq \mu} \dim \mathfrak{g}_\lambda$. The set \mathcal{F} (which is an example of flag variety) carries a natural structure of complex manifold, and the map $w : \mathcal{O}_{\text{ad}(s)} \rightarrow \mathcal{F}$ which sends $u \in \mathcal{O}_{\text{ad}(s)}$ to the filtration (W_u^μ) with $W_u^\mu = \bigoplus_{\lambda \leq \mu} \text{Ker}(\mathbf{i}u - \lambda \text{Id})$ is clearly a diffeomorphism. Let $f : \mathcal{O}_s \rightarrow \mathcal{O}_{\text{ad}(s)}$ be the restriction of $\text{ad} : \mathfrak{k} \rightarrow \text{End } \mathfrak{g}$. We claim that the map

$$\phi = w \circ f : \mathcal{O}_s \rightarrow \mathcal{F}$$

is an immersion and that for any $v \in \mathcal{O}_s$ the image $d\phi(T_v \mathcal{O}_s) \subset T_{\phi(v)} \mathcal{F}$ is a complex subspace, so that there is a unique structure of complex manifold on \mathcal{O}_s with respect to which ϕ is holomorphic. Since for any $h \in K$ we have

$$(5.14) \quad \phi(\text{Ad}(h)(s)) = w(\text{Ad}(h) \text{ad}(s) \text{Ad}(h)^{-1}) = \text{Ad}(h)\phi(s)$$

and the map $\text{Ad}(h) : \mathcal{F} \rightarrow \mathcal{F}$ is a biholomorphism, to prove the claim it suffices to check that $d\phi(s) : T_s \mathcal{O}_s \rightarrow T_{\phi(s)} \mathcal{F}$ is an injection and that its image is invariant under multiplication by \mathbf{i} .

Proving that $d\phi(s)$ is injective is equivalent to proving that $df(s) : T_s \mathcal{O}_s \rightarrow T_{\text{ad}(s)} \mathcal{O}_{\text{ad}(s)}$ is injective, because w is a diffeomorphism. We have $T_s \mathcal{O}_s = \{[a, s] \mid a \in \mathfrak{k}\}$. Assume that $df(s)([a, s]) = 0$. Since f is the restriction of the map $\text{ad} : \mathfrak{k} \rightarrow \text{End } \mathfrak{g}$, which is linear, we deduce from the assumption that $\text{ad}([a, s]) = 0$. Write $a = \sum a_\lambda$, where $a_\lambda \in \mathfrak{g}_\lambda$. Then $[a, s] = -\sum \lambda a_\lambda$, and similarly $0 = \text{ad}([a, s])(s) = [[a, s], s] = \sum \lambda^2 a_\lambda$, so that $a_\lambda = 0$ for each $\lambda \neq 0$. But then we have $[a, s] = -\sum \lambda a_\lambda = 0$. Hence we have proved that $df(s)$ is injective.

We now check that $d\phi(s)(T_s \mathcal{O}_s) \subset T_{\phi(s)} \mathcal{F}$ is a complex subspace. Here we give a direct argument but an alternative and more intrinsic proof of this result may be given using

Lemma 5.4 below. Remark that

$$T_{\phi(s)}\mathcal{F} = \bigoplus_{\lambda < \mu} \text{Hom}(\mathfrak{g}_\lambda, \mathfrak{g}_\mu).$$

Take any $a \in \mathfrak{k}$. By (5.14), $\phi(s)$ sends $[a, s]$ to the projection of $\text{ad}(a) \in \bigoplus_{\lambda, \mu} \text{Hom}(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)$ to $T_{\phi(s)}$. So if we decompose $a = \sum a_\lambda$ as before, then the piece of $d\phi(s)(a)$ in $\text{Hom}(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)$ is $\text{ad}(a_{\mu-\lambda})$ (in particular $d\phi(s)(a)$ only depends on $\sum_{\lambda > 0} a_\lambda$). Denote by $c : \mathfrak{g} \rightarrow \mathfrak{g}$ the conjugation coming from identifying $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, as in the proof of Lemma 5.1. Since \mathfrak{k} is the fixed point set of c and $c(\mathfrak{g}_{-\lambda}) = \mathfrak{g}_\lambda$, it follows from $a \in \mathfrak{k}$ that

$$a' = \sum_{\lambda < 0} -\mathbf{i}a_\lambda + a_0 + \sum_{\lambda > 0} \mathbf{i}a_\lambda \quad \text{also belongs to } \mathfrak{k}.$$

But by the previous observations we have $d\phi(s)([a', s]) = \mathbf{i}d\phi(s)([a, s])$. Hence the image of $d\phi(s)$ is invariant under multiplication by \mathbf{i} , which is what we wanted to prove.

To finish the proof of the lemma, let \mathcal{F}^* be the set of filtrations $(W^\mu)_\mu \in \mathcal{F}$ such that

$$\mathfrak{g} = \bigoplus_{\lambda} \left(W^\lambda \cap \left(\bigoplus_{\nu \geq \lambda} \mathfrak{g}_\nu \right) \right),$$

where the sum runs over the spectrum of $\mathbf{i}\text{ad}(s)$. It is straightforward to check that $\mathcal{F} \setminus \mathcal{F}^*$ is an analytic subvariety of \mathcal{F} . Since $\phi : \mathcal{O}_s \rightarrow \mathcal{F}$ is a holomorphic map and $\mathcal{O}_s^* = \phi^{-1}(\mathcal{F}^*)$, we deduce that $\mathcal{O}_s \setminus \mathcal{O}_s^*$ is an analytic subvariety of \mathcal{O}_s . Finally, since \mathcal{O}_s^* is nonempty (it contains s , for example), $\mathcal{O}_s \setminus \mathcal{O}_s^*$ is not equal to \mathcal{O}_s , and since \mathcal{O}_s is connected this implies that $\dim \mathcal{O}_s \setminus \mathcal{O}_s^* < \dim \mathcal{O}_s$. \square

Lemma 5.4. *Let $\phi : \mathcal{O}_s \rightarrow \mathcal{F}$ be the map defined in the proof of Lemma 5.3. For any $u \in \mathcal{O}_s$ and $g \in G$ we have $\phi(u \cdot g) = \text{Ad}(g^{-1})\phi(u)$.*

Proof. We first state a general result relating morphisms between groups and morphisms between boundaries of the corresponding symmetric spaces. Let $\rho : K \rightarrow K'$ be a morphism of compact connected Lie groups, and denote by the same symbol $\rho : G \rightarrow G'$ the induced map between the complexifications. Let $x_0 = [1_G] \in K \backslash G$ and $x'_0 = [1_{G'}] \in K' \backslash G'$ be the classes of the identity elements. There is a unique map $r : K \backslash G \rightarrow K' \backslash G'$ satisfying $r(x_0 \cdot g) = x'_0 \cdot \rho(g)$ for any $g \in G$. Choosing biinvariant metrics on the Lie algebras of K and K' and taking the induced Riemannian structures on $K \backslash G$ and $K' \backslash G'$, the map r is Lipschitz. Furthermore, r sends geodesic rays in $K \backslash G$ either to constant maps or to geodesic rays in $K' \backslash G'$. More precisely, if $t \mapsto \gamma(t) = [e^{its}g] \in K \backslash G$ is a geodesic ray, then: $t \mapsto r \circ \gamma(t)$ is a geodesic ray in $K' \backslash G'$ unless $d\rho(s) = 0$, in which case we obtain a constant map. Since r is Lipschitz, given any pair of equivalent geodesic rays $\gamma_0 \sim \gamma_1$ either both $r \circ \gamma_0$ and $r \circ \gamma_1$ are geodesic rays or both are constant maps. So one may define the set $\partial_\infty(K \backslash G)^*$ of boundary points corresponding to geodesic rays which are mapped to geodesic rays, and then r induces a continuous map

$$r : \partial_\infty(K \backslash G)^* \rightarrow \partial_\infty(K' \backslash G').$$

It is clear that $\partial_\infty(K \backslash G)^* \subset \partial_\infty(K \backslash G)$ is G -invariant and that r is equivariant, in the sense that for any $y \in \partial_\infty(K \backslash G)^*$ and $g \in G$ we have $r(y \cdot g) = r(y) \cdot \rho(g)$.

Consider the maps $f : \mathcal{O}_s \rightarrow \mathcal{O}_{\text{ad}(s)}$ and $w : \mathcal{O}_{\text{ad}(s)} \rightarrow \mathcal{F}$ given in the proof of Lemma 5.3. We apply the previous observations to the case in which K' is the set of vector space automorphisms of \mathfrak{g} preserving the Hermitian product induced by the biinvariant metric on \mathfrak{k} . The complexification of K' is the group G' of all automorphisms of \mathfrak{g} , and we may take as a morphism $\rho : K \rightarrow K'$ the adjoint representation: $\rho(k) = \text{Ad}(k)$. The conclusion is that $f(u \cdot g) = f(u) \cdot \text{Ad}(g)$ for any $u \in \mathcal{O}_s$ and $g \in G$. Finally, by the results in Section 2.2 we also have $w(u \cdot g) = \text{Ad}(g^{-1})w(u)$ for any $g \in G'$ and $u \in \mathfrak{k}'$ of unit norm. This finishes the proof of the lemma. \square

It follows from the previous lemma that the set $\mathcal{O}_s^* \subset \mathcal{O}_s$ is G -invariant. We next prove that \mathcal{O}_s^* is an orbit of the induced action of $P_{-s} \subset G$ on \mathcal{O}_s .

Lemma 5.5. *The action of P_{-s} on \mathcal{O}_s^* is transitive.*

Proof. Let $\mathfrak{u} = \bigoplus_{\lambda > 0} \mathfrak{g}_\lambda$. This is the Lie algebra of the biggest unipotent subgroup of P_{-s} . Consider the map $e : \mathfrak{u} \rightarrow \mathcal{O}_s^*$ defined as $e(u) = s \cdot e^u$. We are going to prove that the image of e is \mathcal{O}_s^* . Since by Lemma 5.3 \mathcal{O}_s^* is connected, it suffices to prove that $e(\mathfrak{u})$ is open and closed in \mathcal{O}_s^* . From $\mathfrak{p}_s = \bigoplus_{\lambda \leq 0} \mathfrak{g}_\lambda$ we deduce that $\mathfrak{g} = \mathfrak{p}_s \oplus \mathfrak{u}$ which implies, by the implicit function theorem, that any $g \in G$ sufficiently close to 1_G can be written as $g = pe^u$, where $p \in P_s$ and $u \in \mathfrak{u}$. Hence $e(\mathfrak{u})$ is open in $P_s \backslash G$ (use the same arguments as in the proof of Lemma 5.1). Let the map $\phi : \mathcal{O}_s \rightarrow \mathcal{F}$ and the subset $\mathcal{F}^* \subset \mathcal{F}$ be those defined in the proof of Lemma 5.3. We then have $\phi(\mathcal{O}_s^*) \subset \mathcal{F}^*$. There is a biholomorphism

$$\gamma : \mathcal{F}^* \rightarrow \bigoplus_{\lambda < \mu} \text{Hom}(\mathfrak{g}_\lambda, \mathfrak{g}_\mu),$$

characterized by the property that γ^{-1} sends $\delta = (\delta_{\lambda\mu}) \in \bigoplus_{\lambda < \mu} \text{Hom}(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)$ to the filtration $(W^\mu(\delta))_\mu$ in which $W^\mu(\delta) = \bigoplus_{\xi \leq \mu} \text{Graph}(\delta_\xi)$, where $\delta_\xi = \sum_{\mu \geq \xi} \delta_{\xi\mu} : \mathfrak{g}_\xi \rightarrow \bigoplus_{\mu \geq \xi} \mathfrak{g}_\mu$. Now, to check that $e(\mathfrak{u}) \subset \mathcal{O}_s^*$ is closed it suffices to prove, similarly to Lemma 5.1, that the map f defined as the following composition is proper:

$$f : \mathfrak{u} \xrightarrow{e} \mathcal{O}_s^* \xrightarrow{\phi} \mathcal{F}^* \xrightarrow{\gamma} \bigoplus_{\lambda < \mu} \text{Hom}(\mathfrak{g}_\lambda, \mathfrak{g}_\mu).$$

Let $0 < \lambda_1 < \dots < \lambda_r$ be the positive eigenvalues of $\mathbf{i} \text{ad}(s)$. For any $u = \sum_{\lambda_i} u_{\lambda_i} \in \mathfrak{u}$ write $u_i := u_{\lambda_i}$, and let $f(u)_j$ denote the component of $f(u)$ in $\text{Hom}(\mathfrak{g}_0, \mathfrak{g}_{\lambda_j})$. We deduce from the definitions that $f(u) = \exp(-\text{ad}(u)) - 1$ for any $u \in \mathfrak{u}$. Since by the Jacobi identity $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$, and the decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ is finite, there exist polynomials P_j such that $P_j(0, \dots, 0) = 0$ and

$$(5.15) \quad f(u)_j = \text{ad}(u_j)_j + P_j(\text{ad}(u_1), \dots, \text{ad}(u_{j-1})),$$

where $\text{ad}(u_j)_j$ denotes the piece of $\text{ad}(u_j) \in \text{End } \mathfrak{g}$ in $\text{Hom}(\mathfrak{g}_0, \mathfrak{g}_{\lambda_j})$.

Consider the Hermitian norm on \mathfrak{g} induced by the biinvariant norm on \mathfrak{k} and define, for any $\alpha \in \text{End } \mathfrak{g}$, $|\alpha| = \sup |\alpha(v)|/|v|$, where the supremum runs over the set of nonzero

$v \in \mathfrak{g}$. Given any $u \in \mathfrak{u}$, we have for each j :

$$(5.16) \quad |\mathrm{ad}(u_j)_j| \geq |\mathrm{ad}(u_j)(s)|/|s| = \lambda_j |u_j|.$$

Let $f(u)_j$ be the piece of $f(u)$ in $\mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{\lambda_j})$. Then we have $f(u)_{\lambda_j} = \mathrm{ad}(u_{\lambda_j})$, so

$$|f(u)| \geq |f(u)_{\lambda_j}| \geq |f(u)_{\lambda_j}(s)|/|s| = |[u_{\lambda_j}, s]|/|s| = \lambda_j |u_{\lambda_j}|.$$

There exist polynomials $p_j \in \mathbb{C}[t]$ vanishing at $t = 0$ such that

$$|P_j(\mathrm{ad}(u_1), \dots, \mathrm{ad}(u_{j-1}))| \leq p_j(|u_1| + \dots + |u_{j-1}|)$$

for each j and u_1, \dots, u_{j-1} . Since $p_1(0) = \dots = p_r(0) = 0$, there exists an $\epsilon > 0$ such that the following system of inequalities

$$\begin{aligned} t_1 &< \epsilon(t_1 + \dots + t_r) \\ t_2 &< \epsilon(t_1 + \dots + t_r) + 2\lambda_2^{-1}p_2(t_1) \\ &\vdots \\ t_r &< \epsilon(t_1 + \dots + t_r) + 2\lambda_r^{-1}p_r(t_1 + \dots + t_{r-1}) \end{aligned}$$

has no solution (t_1, \dots, t_r) satisfying $t_j \geq 0$ for each j . Let us prove that for any $u \in \mathfrak{u}$ we have $|f(u)| \geq \lambda_1 \epsilon |u|/2$, which clearly implies that f is proper. Define $t_j = |u_j|$ for each j . By the choice of ϵ at least one of the previous inequalities does not hold, say the j -th one. Then we have (setting $p_1 = 0$ and $P_1 = 0$ when $j = 1$)

$$|u_j| \geq \epsilon(|u_1| + \dots + |u_r|) + 2\lambda_j^{-1}p_j(|u_1| + \dots + |u_{j-1}|)$$

which implies, using (5.16) and the definition of p_j ,

$$|\mathrm{ad}(u_j)_j| \geq \lambda_j |u_j| \geq 2p_j(|u_1| + \dots + |u_{j-1}|) \geq 2|P_j(\mathrm{ad}(u_1), \dots, \mathrm{ad}(u_{j-1}))|.$$

Combining this with (5.15) we obtain

$$\begin{aligned} |f(u)| &\geq |f(u)_j| \geq |\mathrm{ad}(u_j)_j|/2 \geq \lambda_j |u_j|/2 \\ &\geq \lambda_j \epsilon (|u_1| + \dots + |u_r|)/2 \\ &\geq \lambda_j \epsilon |u|/2 \geq \lambda_1 \epsilon |u|/2. \end{aligned}$$

This finishes the proof of the lemma. \square

5.3. Proof of Lemma 2.3. Assume that e_u, e_v are geodesically connected, and let $\gamma : \mathbb{R} \rightarrow K \backslash G$ be a geodesic such that $\gamma(t) \rightarrow e_u$ when $t \rightarrow \infty$ and $\gamma(t) \rightarrow e_v$ when $t \rightarrow -\infty$. By Lemma 5.1 there exists some $h \in P_u$ such that $\gamma(0) \cdot h = x_0$. Since $\gamma \cdot h$ is a geodesic passing through x_0 at time 0, it is of the form $\gamma \cdot h(t) = e^{its}$, and since $\gamma \cdot h(t)$ converges to e_u as $t \rightarrow \infty$, we have $s \cdot h = s = u$. Then $v \cdot h = -u$. By Lemma 5.2, v belongs to the adjoint orbit $\mathcal{O}_{-u} \subset \mathfrak{k}$. Obviously the endomorphisms $\mathrm{ad}(u), \mathrm{ad}(-u) \in \mathrm{End} \mathfrak{g}$ are opposed (in the sense specified in the Introduction), and Lemma 5.4 implies that $\mathrm{ad}(u), \mathrm{ad}(v)$ are opposed as well. Hence, u and v are opposed.

Conversely, assume that u and v are opposed. Then $v \in \mathcal{O}_{-u}^*$. By Lemma 5.5 the action of P_u on \mathcal{O}_{-u}^* is transitive, so there exists some $h \in P_u$ such that $v \cdot h = -u$.

Then the geodesic $\gamma(t) = [e^{itu}h^{-1}]$ satisfies $\gamma(t) \rightarrow e_u$ when $t \rightarrow \infty$ and $\gamma(t) \rightarrow e_v$ when $t \rightarrow -\infty$, so e_u and e_v are geodesically connected. This finishes the proof of the Lemma.

REFERENCES

- [B] W. Ballmann, *Lectures on spaces of nonpositive curvature, With an appendix by Misha Brin* DMV Seminar **25**, Birkhuser Verlag, Basel, 1995.
- [E] P. Eberlein, Structure of manifolds of nonpositive curvature, *Global differential geometry and global analysis 1984* (Berlin, 1984), 86–153, Lecture Notes in Math. **1156** Springer, Berlin, 1985.
- [GS] V. Guillemin, S. Sternberg, Geometric Quantization and Multiplicities of Group Representations, *Invent. Math.* **67** (1982) 515–538.
- [HH] P. Heinzner, A.T. Huckleberry, Kählerian structures on symplectic reductions, *Complex Analysis and Algebraic Geometry*, T. Peternell, F.-O. Schreyer eds., W. de Gruyter, 2000.
- [MFK] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, 3rd edition, Erg. Math., Springer Verlag (1994).
- [M] I. Mundet i Riera, A Hitchin-Kobayashi correspondence for Kaehler fibrations, *J. Reine Angew. Math.* **528** (2000), 41–80.
- [Sch] G. Schwarz, The topology of algebraic quotients, in *Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988)*, 135–151, Progr. Math. **80** Birkhuser Boston (1989).
- [SL] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, *Ann. of Math. (2)* **134** (1991), no. 2, 375–422.
- [S] R. Sjamaar, Holomorphic slices, symplectic reduction and multiplicities of representations, *Ann. Math. II*, **141** No. 1 (1995) 87–129.
- [T] A. Teleman, Symplectic stability, analytic stability in non-algebraic complex geometry, *Internat. J. Math.* **15** (2004), no. 2, 183–209.

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